

Gauge-free Electrodynamics

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Abstract

We propose a reformulation of electrodynamics in terms of a *physical* vector potential entirely free of gauge ambiguities. Quantizing the theory leads to a propagator that is gauge invariant by construction in this reformulation, in contrast to the standard photon propagator. Coupling the theory to a charged Abelian Higgs field leads at the quantum level to a one loop effective potential which realizes the Coleman-Weinberg mechanism of mass generation, thus resolving the issue of its gauge dependence. We relate our results to recent work by Niemi et. al. and Faddeev, where similar strategies are adopted in a version of the electroweak theory. Other theories with linear Abelian gauge invariance, like the linearized spin 2 theory of gravity or the antisymmetric tensor field, which may all be reformulated in terms of physical vector or tensor potentials without gauge ambiguities, are also discussed briefly.

1 Introduction

The infinite gauge redundancy associated with a gauge potential in gauge theories is frequently interpreted as a sign of its unphysical nature. Usually, holonomies of the electrodynamic gauge potential over non-contractible loops in spacetime, associated with the Ehrenberg-Siday-Aharonov-Bohm [1] phase in quantum mechanics, are thought of as its only physical manifestation, apart from its appearance in derivative form in the electromagnetic field strength (curvature) tensor. Is it possible to reformulate electrodynamics in terms of an entirely physical, local vector potential (without any gauge ambiguities), such that all consequences of standard electrodynamics emerge from it, without ever having to worry about *gauge fixing* ? How would such a theory be different from a gauge-fixed version of standard electrodynamics ? These are the issues that we seek to address in this paper. The main idea here is the demonstration that gauge fixing in electrodynamics can be avoided by formulating the theory using gauge invariant local potentials, as opposed to

local field strengths. Even though arrived at independently, the basic idea in this paper overlaps with those of Niemi et. al. [2] and Faddeev [3] (which appeared prior to this paper), who have shown that the $SU(2)$ gauge redundancy in a version of the electroweak theory (without a Higgs potential) is removed in the Higgs boson-vector boson sector, at the very outset, by a clever choice of variables, rather than in the usual roundabout process of fixing gauge, leaving behind a $U(1)$ gauge theory with a scalar field coupled to charged vector bosons. In this paper we show that even when scalar fields are not available, at least in Abelian theories like electrodynamics, a choice of variables to a gauge-free formalism is indeed possible. However, as we shall argue later, the interpretation in [3] of the scalar field as the conformal mode in a conformally flat spacetime is true only at the tree level, and does not hold when perturbative effects are taken into account, as in the one loop effective potential.

We review in the next section the standard gauge redundancy in vacuum electrodynamics, just to agree on notations. In the same section we discuss a spacetime transverse projection of the standard 4-vector potential which is manifestly gauge invariant. Using this as a cue, we propose, in the next section, a 4-vector potential field obeying a spacetime divergenceless condition as a *physical* condition. This condition is to be thought of as akin to the spatial divergencelessness of magnetic fields in standard electrodynamics and is certainly related to the absence of isolated magnetic charges in Nature. It is *not* a gauge fixing condition. Indeed, unlike the standard formulation, the theory is not described in terms of field strengths. Maxwell's equations are shown to *emerge* from field equations for the photon field coupled to a conserved current. We then quantize the non-interacting theory and derive the free photon propagator. In Section V, we couple the theory to charged Higgs fields and study the Coleman-Weinberg perturbative mass generation mechanism. We show that the interpretation of the 'radial' component of Abelian Higgs field as a dilaton is precluded by cancellation with terms arising in the one loop effective potential. In the next section, we briefly discuss the free linearized spin 2 theory of gravity and free antisymmetric tensor field theory in four dimensional Minkowski spacetime, as examples where gauge-free formulations are indeed possible. We conclude in Section VII with some surmises about how one may proceed to formulate Yang Mills theory in terms of physical potential fields alone.

2 Maxwell electrodynamics

2.1 Gauge Redundancy

The standard formulation of Maxwell electrodynamics in four dimensional Minkowski spacetime entails introducing a one form gauge potential \mathbf{A} and a two form field strength \mathbf{F} defined as $\mathbf{F} \equiv d \mathbf{A}$. Clearly, the Bianchi identity

$$d \mathbf{F} = 0 \tag{1}$$

holds.

Introducing a smooth one form current density source \mathbf{J} , the action for Maxwell electrodynamics takes the form

$$S[\mathbf{A}] = \int_{\mathcal{M}} [\mathbf{F} \wedge^* \mathbf{F} + \mathbf{A} \wedge^* \mathbf{J}] , \quad (2)$$

where, $^* \mathbf{F}$ is the Hodge-dual of \mathbf{F} , leading to the field equations of motion

$$\tilde{d} \mathbf{F} = - \mathbf{J} \quad (3)$$

where, $\tilde{d} \equiv (-)^{Dp+p+1} d^*$ is the *co-differential* which maps p -forms to $p - 1$ forms on spacetime. The above equation, when expressed in terms of \mathbf{A} , assumes the form

$$\square \mathbf{A} - d \tilde{d} \mathbf{A} = - \mathbf{J} . \quad (4)$$

where, $\square \equiv d\tilde{d} + \tilde{d}d$ is the d'Alembertian or the Laplacian in four dimensional Minkowski spacetime. In terms of Cartesian components (in a flat spacetime with the Minkowski metric having the signature $diag(1, -1, -1, -1)$), eq. (4) becomes

$$\square A_\mu - \partial_\mu \partial_\nu A^\nu = - J_\mu . \quad (5)$$

The solution of eq. (4) (or, equivalently, of (5)) is of course singular because of the infinite ambiguity under $U(1)$ gauge transformations :

$$\mathbf{A} \rightarrow \mathbf{A}^{(\omega)} = \mathbf{A} + d \omega \quad (6)$$

for any smooth function ω on spacetime.

The singularity is avoided by ‘gauge-fixing’; a popular choice is the class of gauges called Lorentz gauges: $\tilde{d}\mathbf{A} = 0$, which reduces (3) to the inhomogeneous wave equation $\square \mathbf{A} = -\mathbf{J}$. But there are countless other choices for gauge fixing. Since the formulation appears to yield a gauge potential based on our choice (rather than Nature’s), we are accustomed to considering it as being *unphysical*, and tend to rely only on gauge invariant quantities constructed out of it, like field strengths, Wilson loops etc. It stands to reason though that the gauge potential cannot entirely be unphysical if it has to yield physical quantities upon differentiation or integration. The question then reduces to the following: what part of the gauge potential contains all the physics ? If this part of it could be extracted and used as the basis of formulating the theory, it ought to be free of the ambiguities alluded to above. This is what we attempt to achieve in the next section.

2.2 Transverse Projection

We begin with a $U(1)$ gauge one form \mathbf{A} , and look for a projection operator \mathcal{P} with the following properties:

- $[\mathcal{P}]^2 = \mathcal{P}$.
- $\mathcal{P} \mathbf{A}$ is a one-form field on flat spacetime.
- Under a gauge transformation $\mathcal{P} \mathbf{A} \rightarrow [\mathcal{P} \mathbf{A}]^{(\omega)} = \mathcal{P} \mathbf{A}$.

Such a projection operator should project out for us the part of the gauge potential that exists in Nature.

Since the theory is Abelian, the third condition above can be reduced to $\mathcal{P} \mathbf{A}^\omega = \mathcal{P} \mathbf{A}$. With $\mathbf{A}^\omega = \mathbf{A} + d\omega$ the projection operator must satisfy

$$\mathcal{P} d\omega = 0 . \quad (7)$$

A solution of eq. (7) for \mathcal{P} may be given as

$$\mathcal{P} = \tilde{d} \square^{-1} d = 1 - d \square^{-1} \tilde{d} . \quad (8)$$

If $\tilde{\mathcal{P}} \equiv 1 - \mathcal{P}$, then \mathbf{A}_T clearly satisfies the constraint

$$\tilde{\mathcal{P}} \mathbf{A}_T = 0 . \quad (9)$$

In a Cartesian basis, the projection operator takes the more familiar form

$$\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \square^{-1} \partial_\nu \quad (10)$$

which, acting on \mathbf{A} is expressed more explicitly as

$$A_{T,\mu} = \mathcal{P}_{\mu\nu} A^\nu = A_\mu - \partial_\mu \int d^4x' \mathcal{G}(x-x') \partial'_\nu A^\nu(x') \quad (11)$$

$$\equiv A_\mu - \partial_\mu a(x) , \quad (12)$$

where $\mathcal{G}(x-x')$ is the Green's function of the d'Alembertian operator. Eq. (14) translates into the transversality condition in spacetime

$$\partial_\mu \mathbf{A}_T^\mu = 0 . \quad (13)$$

The spacetime longitudinal mode $a(x)$ carries the entire gauge transformation, transforming as $a(x) \rightarrow a(x) + \omega(x)$, which also testifies to its unphysicality.

3 Gauge-free Reformulation

Rather than project out the unphysical longitudinal part of vector potential in Maxwell electrodynamics by fixing gauge, we attempt here an *alternative* formulation where the

fundamental field degree of freedom has *no* unphysical part. In other words, we begin by *defining* a one form field $\mathbf{A}_{\mathcal{P}}$ on Minkowski spacetime, which is *co-closed*

$$\tilde{d} \mathbf{A}_{\mathcal{P}} = 0 , \quad (14)$$

or equivalently, a vector field which obeys the spacetime constraint of vanishing 4-divergence

$$\partial_{\mu} \mathbf{A}_{\mathcal{P}}^{\mu} = 0 . \quad (15)$$

We further propose that the one form field is described by the action

$$S_{gf} = \int [- \frac{1}{2} * \mathbf{A}_{\mathcal{P}} \wedge \square \mathbf{A}_{\mathcal{P}} + * \mathbf{J} \wedge \mathbf{A}_{\mathcal{P}}] , \quad (16)$$

where \mathbf{J} is a co-closed one form source. It is easy to derive the equation of motion for $\mathbf{A}_{\mathcal{P}}$

$$\square \mathbf{A}_{\mathcal{P}} = - \mathbf{J} , \quad (17)$$

which has the solution (modulo boundary conditions)

$$\mathbf{A}_{\mathcal{P}} = - \square^{-1} \mathbf{J} , \quad (18)$$

or, in terms of Cartesian components

$$A_{\mathcal{P}}^{\mu}(x) = - \int d^4 x' \mathcal{G}(x - x') J^{\mu}(x') . \quad (19)$$

How unique is this ‘photon’ field $\mathbf{A}_{\mathcal{P}}$, defined as above ? We notice that indeed, this field does not admit the full $U(1)$ group of local gauge transformations that the standard electrodynamic gauge potential does. However, there is a residual ambiguity: the divergencefree constraint (and also the action) remains invariant under the transformation $\mathbf{A}_{\mathcal{P}} \rightarrow \mathbf{A}_{\mathcal{P}} + d\omega$ for ω obeying $\square\omega = 0$ everywhere. It is obvious though that this ambiguity is of no consequence for the field degrees of freedom coupled to the source. It will at most affect the *homogeneous* solution which is determined solely by boundary conditions. If we require that ω is subject to boundary conditions so that it reduces to a constant at asymptopia, then the only globally harmonic function on Minkowski space is a constant everywhere. This means that $\mathbf{A}_{\mathcal{P}}$ is effectively unique in spacetime and hence can well represent the photon field.

The Maxwell equations can now be shown to follow from this photon field: The Maxwell field strength two form \mathbf{F} is *defined* in the usual way as $\mathbf{F}(\mathbf{A}_{\mathcal{P}}) \equiv d \mathbf{A}_{\mathcal{P}}$. The Bianchi identity $d \mathbf{F}(\mathbf{A}_{\mathcal{P}}) = 0$ follows immediately. Using eqn.s (15) and (17), one immediately derives the Maxwell equations with conserved sources,

$$\tilde{d} F = - J , \quad (20)$$

or, equivalently, in terms of Cartesian components,

$$\partial_\mu F^{\mu\nu} = -J^\nu . \quad (21)$$

The most general solution for \mathbf{F} can be obtained from eq. (19) through appropriate derivatives. Thus, the vector field $\mathbf{A}_{\mathcal{P}}$ is the single entity from which electric and magnetic fields can be *derived* by taking appropriate spacetime derivatives. $\mathbf{A}_{\mathcal{P}}$ is as physical as electric and magnetic fields are in the standard Maxwell formulation. One may claim, as already mentioned in the Introduction, that the unification of electricity and magnetism which was envisioned by Maxwell is actually realized explicitly through $\mathbf{A}_{\mathcal{P}}$ which thus captures the entire essence of electrodynamics.

One question that may be asked about the reformulation of electrodynamics in this section is that it appears to be similar to a gauge-fixed version of standard electrodynamics, gauge-fixed to the gauge $\partial \cdot \mathbf{A} = 0$. The difference is in the classical action (16), which differs from the standard Maxwell action in that the latter is gauge invariant under full $U(1)$ gauge transformations, being a functional of field strengths. In contrast, the action (16) is entirely a functional of the *physical* vector potential $\mathbf{A}_{\mathcal{P}}$. The similarity with a gauge fixed version of the standard formulation is only formal; there is nothing physical about gauge fixing, since it is a way of constraining the unphysical degrees of freedom in the conventional formulation. In contrast, there is, in the gauge-free approach, no unphysical degree of freedom to fix.

Is there a direct way to experimentally infer the existence of the physical vector potential introduced in this section ? We do not know the answer to this yet.

3.1 Source-free Gauge-free electrodynamics : Classical formulation

Consider now the vacuum sector corresponding to $\mathbf{J} = 0$; Eq. (17) reduces to the homogeneous wave equation

$$\square \mathbf{A}_{\mathcal{P}} = 0 \quad (22)$$

which has plane wave like solutions

$$\mathbf{A}_{\mathcal{P}} = \Re \mathcal{A} \exp i\psi , \quad (23)$$

where, we assume that the amplitude \mathcal{A} is a slowly-varying one form field on spacetime, while the phase function ψ varies much more rapidly; in other words, schematically

$$|\partial_\mu \psi| \gg |\partial_\mu \mathcal{A}| \quad (24)$$

for all components. Substituting the ansatz (23) into (15) and (22), and defining $\mathbf{k} \equiv -id \psi$ one obtains in the approximation (with slight abuse of notation)

$$\mathbf{k} \cdot \mathbf{A}_{\mathcal{P}} = 0 = k^2 \quad (25)$$

Thus $\mathbf{A}_{\mathcal{P}}$ as a vector field is orthogonal in spacetime to the null vector field \mathbf{k} . As we argue below, this has the important consequence that $\mathbf{A}_{\mathcal{P}}$ must be spacelike everywhere, or null in regions of spacetime where it is parallel to \mathbf{k} .

Consider the null tetrad basis of flat spacetime, given by the four null vectors \mathbf{l} , \mathbf{n} which are real, and the complex conjugate pair \mathbf{m} , $\bar{\mathbf{m}}$, obeying the conditions

$$\mathbf{l} \cdot \mathbf{n} = 1, \quad \mathbf{m} \cdot \bar{\mathbf{m}} = -1 \quad (26)$$

with all other scalar products vanishing. Without any loss of generality, the null tetrad basis may be so chosen that the basis vector \mathbf{n} is parallel with the wave vector \mathbf{k} defined above. Expanding the transverse potential $\mathbf{A}_{\mathcal{P}}$ in this basis

$$\mathbf{A}_{\mathcal{P}} = A_{\mathcal{P}}^{(n)} \mathbf{l} + A_{\mathcal{P}}^{(l)} \mathbf{n} - A_{\mathcal{P}}^{(\bar{m})} \mathbf{m} - A_{\mathcal{P}}^{(m)} \bar{\mathbf{m}}, \quad (27)$$

the first of the equations (25) immediately implies that the component $A_{\mathcal{P}}^{(n)} = \mathbf{n} \cdot \mathbf{A}_{\mathcal{P}} = 0$. Consequently, the squared norm of $\mathbf{A}_{\mathcal{P}}$ (considered as a vector field in spacetime) is given by

$$\mathbf{A}_{\mathcal{P}}^2 = -2|A_{\mathcal{P}}^{(m)}|^2 < 0, \quad (28)$$

implying that this potential is everywhere spacelike. Furthermore, *the integral curves of the potential 4 vector field are closed curves in spacetime*. It is quite remarkable that even though this was not put in, there are no closed *timelike* integral curves of the vector potential.

Clearly, the component $A_{\mathcal{P}}^{(l)}$, even though not necessarily zero in an arbitrary Lorentz frame, may indeed be made to vanish by a suitable choice of a frame. In this special frame, the only non-vanishing components are the complex combination $A_{\mathcal{P}}^{(m)}$. It is now natural to identify these components with the two spatial degrees of transverse polarization that electromagnetic waves in vacuum are known to have. These correspond in the quantum description to the two helicity states of helicity ± 1 of a massless vector field in Minkowski spacetime. Note that the existence of the two spatial polarizations does not involve any gauge choice; it does however involve a choice of a Lorentz frame. One further note of interest is that the complex polarization $A_{\mathcal{P}}^{(m)}$ basically spans a 2-sphere (for free photon fields of a given intensity) which may be identified with the Poincaré sphere.

3.2 Source-free Gauge-free Electrodynamics : Functional Quantization

The quantum theory of a free transverse vector field can arrived at using the functional integral formulation. In contrast to the standard approach where the Faddeev-Popov prescription is usually employed to integrate out the gauge degrees of freedom, no gauge fixing need be employed here. However, since the functional integral describing the vacuum-to-vacuum amplitude is over all configurations of the vector field $\mathbf{A}_{\mathcal{P}}$, the transversality

constraint (15) must be directly inserted into the integral to ensure that the integral is only over transverse field configurations. Recall that this is *not* a gauge condition; it is instead a constraint defining the physical photon degrees of freedom.

The relevant vacuum-to-vacuum amplitude (in presence of a transverse source) is given by

$$\begin{aligned} Z[\mathbf{J}] &= \int \mathcal{D}\mathbf{A}_{\mathcal{P}} \exp i \left(\frac{1}{2} A_{\mathcal{P}}^{\mu} \square A_{\mathcal{P},\mu} + \int d^4x \mathbf{J} \cdot \mathbf{A}_{\mathcal{P}} \right) \delta[\partial_{\mu} \mathbf{A}_{\mathcal{P}}^{\mu}] \\ &= \int \mathcal{D}\mathcal{S} \mathcal{D}\mathbf{A}_{\mathcal{P}} \exp i \int d^4x \left[\frac{1}{2} A_{\mathcal{P}}^{\mu} \square A_{\mathcal{P},\mu} + (J_{\mu} - \partial_{\mu} \mathcal{S}) A_{\mathcal{P}}^{\mu} \right]. \end{aligned} \quad (29)$$

In the second line of (29) we have introduced an auxiliary scalar field \mathcal{S} which acts as the Lagrange multiplier for the physical constraint (15). The integral over $\mathbf{A}_{\mathcal{P}}$ is easily done, yielding

$$\begin{aligned} Z[\mathbf{J}] &= (\det \eta_{\mu\nu} \square)^{-1/2} \int \mathcal{D}\mathcal{S} \exp \frac{-i}{2} \int d^4x d^4y [\partial_{\mu}^x \mathcal{S}(x) - J_{\mu}(x)] \\ &\quad \cdot \mathcal{G}(x-y) \eta^{\mu\nu} [\partial_{\nu}^y \mathcal{S}(y) - J_{\nu}(y)], \end{aligned} \quad (30)$$

where, $\mathcal{G}(x-y)$ is the d'Alembertian Green's function. A series of partial integrations and using the transversality of the current density \mathbf{J} , and also identities like $\partial_x \mathcal{G}(x-y) = -\partial_y \mathcal{G}(x-y)$ leads to the simple expression

$$\begin{aligned} Z[\mathbf{J}] &= (\det \eta_{\mu\nu} \square)^{-1/2} \exp -\frac{1}{2} i \int d^4x d^4y J^{\mu}(x) \Delta_{\mu\nu}(x-y) J^{\nu}(y) \\ &\quad \cdot \int \mathcal{D}\mathcal{S} \exp -\frac{i}{2} \int d^4x \mathcal{S}^2(x), \end{aligned} \quad (31)$$

where, $\Delta_{\mu\nu}(x-y) \equiv \eta_{\mu\nu} \mathcal{G}(x-y)$. The integral over \mathcal{S} is a trivial Gaussian producing a constant (albeit infinite) independent of \mathbf{J} . It will thus cancel out, along with the prefactor $\det^{-1/2} \square$ in eq. (30) from the generating functional for connected Green's functions $W[\mathbf{J}] = -i \log Z[\mathbf{J}]$, and all vacuum expectation values.

It is now straightforward to extract the free photon propagator from eq. (31):

$$\mathcal{G}_{\mu\nu}(x-y) \equiv \frac{1}{2} \frac{\delta^2 W[\mathbf{J}]}{\delta J^{\mu}(x) \delta J^{\nu}(y)} \Big|_{\mathbf{J}=0} \quad (32)$$

$$= \eta_{\mu\nu} \mathcal{G}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x-y)} \frac{i \eta_{\mu\nu}}{p^2 + i\epsilon}. \quad (33)$$

Clearly, this propagator does not possess any gauge artifacts.

We close this section with an observation: even though the physical photon field is spacetime transverse, its propagator is *not* the same as the *Landau* gauge propagator of

standard QED. It is what in the standard formulation would correspond to the *Feynman* gauge. This is different from the Landau-Lorentz gauge condition which formally resembles our divergence free constraint (15). In the standard approach, one adds to the free Maxwell action the gauge fixing term $(1/2\alpha)(\partial \cdot \mathbf{A})^2$ corresponding to the so-called α -gauges. The Feynman gauge corresponds to the choice $\alpha = 1$ and the Lorentz-Landau gauge to the limit $\alpha \rightarrow 0$. In our gauge-free formulation, spacetime divergencelessness is *not* a matter of choice, it is a defining feature of what we mean by electromagnetism.

4 Gauge-free Electrodynamics with Sources

4.1 Preliminaries

Charged matter fields by themselves are described by theories invariant under *global* gauge $U(1)$ transformations which are phase transformations on a complex fields $\Phi \rightarrow e^{i\omega}\Phi$ for ω a real constant. This implies that there is choice of field basis in which the only transformation is that of some real ‘phase’ fields undergoing the same constant shift : $\theta(x) \rightarrow \theta^{(\omega)}(x) = \theta(x) + \omega$. This means that in this field basis, the Lagrange density must depend on these phase fields only through their derivatives. Now, when we *gauge* this $U(1)$, i.e., make $\omega = \omega(x)$ which is an arbitrary function of \mathbf{x} , it is obvious that the transformation of the phase fields is such that some of them lose their physicality: $\theta^{(\omega(x))}(x) = \theta(x) + \omega(x)$. E.g., for charged scalar fields, there is a single phase field $\theta(x)$ which is thus rendered unphysical, since one can change it as one pleases by choosing the function $\omega(x)$. The choice $\omega(x) = -\theta(x)$ ‘gauges away’ $\theta(x)$. Demanding that $S[\rho^\omega, \theta^\omega] \equiv \tilde{S}[\rho, \theta, \omega] = S[\rho, \theta]$ is impossible, as is well-known. In the standard formulation, one changes the action by adding gauge potentials such that the phase fields occur only through the ω -*independent* combination $\Theta(x) \equiv \theta(x) - a(x)$ where the longitudinal part $a(x)$ of the standard gauge potential was introduced in eq. (12). Once this is done, the entire action is written exclusively in terms of *physical* fields on which local gauge transformations are inert. In this sense, gauge invariance is not a *symmetry* of the theory, but merely a statement that a gauge invariant theory can be described by an action which is a functional *exclusively of gauge invariant dynamical variables*.

4.2 Abelian Higgs Model

In the conventional formulation of the Abelian Higgs model [4], [5], the gauge potential \mathbf{A} couples to the Nöther current of the charged scalar field corresponding to global $U(1)$ invariance. In addition, there is a contact interaction (the ‘seagull’) dictated by local $U(1)$ *gauge* invariance. In the gauge-free approach, one begins instead with the action (suppressing obvious indices)

$$S[\rho, \Theta, \mathbf{A}_P] = \int d^4x \left[\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}e^2\rho^2(\mathbf{A}_P - e^{-1}\partial\Theta)^2 - \frac{1}{4}F_{\mu\nu}(\mathbf{A}_P)F^{\mu\nu}(\mathbf{A}_P) - V(\rho) \right], \quad (34)$$

where $V(\rho)$ is the scalar potential, and $\mathbf{A}_{\mathcal{P}}$ obeys the transversality constraint (15). As alluded above, all fields in this action are explicitly invariant under $U(1)$ gauge transformations. It is interesting that the phase field Θ occurs in the action only through the combination $\mathbf{A} - e^{-1}d\Theta$; this implies that the shift $\Theta \rightarrow \Theta + \text{const.}$ is still a symmetry of the action for all constant shifts. However, since there is no canonical kinetic energy term for Θ , it is hard to associate a propagating degree of freedom with Θ . Instead, one can interpret it as an external field with derivative interactions with the vector field $\mathbf{A}_{\mathcal{P}}$ and the Higgs field ρ .

One can now think of two kinds of scalar potentials $V(\rho)$: one for which the minimum of the potential $\langle \rho \rangle = 0$ and the other for which the minimum lies away from the origin $\langle \rho \rangle = \rho_0 \neq 0$. It is this second case which is of interest to us. One first makes a field redefinition

$$\mathbf{Y} \equiv \mathbf{A}_{\mathcal{P}} - e^{-1} d\Theta . \quad (35)$$

Observe that the vector field \mathbf{Y} is explicitly gauge invariant even under the restricted gauge transformations that one may wish to subject $\mathbf{A}_{\mathcal{P}}$ to. In any case, the action is now a functional of ρ and \mathbf{Y} alone; the Θ field has disappeared from it. This happens irrespective of whether the scalar potential $V(\rho)$ has a minimum away from the origin or not:

$$S[\rho, \mathbf{Y}] = \int d^4x \left[\frac{1}{2}(\partial\rho)^2 - V(\rho) - \frac{1}{4}F_{\mu\nu}(\mathbf{Y})F^{\mu\nu}(\mathbf{Y}) + \frac{1}{2}e^2\rho^2 Y^2 \right] \quad (36)$$

If $V(\rho)$ has a minimum at $\rho = \rho_0 \neq 0$ one now also defines $\rho \rightarrow \rho + \rho_0$, it is easy to see that the \mathbf{Y} acquires a mass $m_{\mathbf{Y}}^2 = e^2\rho_0^2$ while the ρ also acquires a mass $m_{\rho}^2 = V''(\rho_0)$. Observe that the field redefinition (35) is no longer a gauge transformation because it involves physical degrees of freedom.

The essential aspect of the Higgs mechanism is a mechanism of *hiding* the *global* $U(1)$ symmetry of the original charged scalar field theory, through its interaction with the physical Maxwell vector potential [6]. In doing so, the new vector potential \mathbf{Y} is no longer subject to the transversality constraint (15). It thus has one degree of freedom more than the $\mathbf{A}_{\mathcal{P}}$. But the \mathbf{Y} field has been constructed by $\mathbf{A}_{\mathcal{P}}$ ‘eating’ a *physical* degree of freedom, the Θ field, and not an unphysical one which may be a gauge artifact.

4.3 Gauge-free scalar QED: Coleman-Weinberg Mechanism

The Coleman-Weinberg mechanism is a radiative mechanism whereby a scalar electrodynamics theory with massless photons and scalar bosons changes its spectrum due to perturbative quantum corrections. Both the neutral component of the scalar boson and the vector boson acquire physical masses given by the parameters of the theory. In its incipient formulation, the mechanism has been shown to be gauge-dependent, thereby

casting doubt on its physicality. Using the gauge free reformulation given above, we compute in this section the one loop effective potential of the theory, and argue that the effect is physical at this level.

The action for the theory is already given above (eq. (34)), with the choice $V(\rho) = (\lambda/4!)\rho^4$. Following [7], the theory is quantized using the functional integral formalism. In the standard formulation of QED, one needs to resort to the Faddeev-Popov technique of gauge fixing and extracting the infinite volume factor associated with the group of gauge transformations, from the vacuum persistence amplitude (generating functional for all Green's functions), in order that this amplitude does not diverge upon integrating over gauge equivalent copies of the gauge potential. In the gauge free approach here, this technique is not necessary. The integration over the transverse gauge potential is, of course, restricted to configurations that obey the spacetime transversality condition (15). Since the integration variables are unambiguous, the task, at least at the one loop level, is simpler.

The generating functional is thus given by

$$Z[J, J', \mathbf{J}] = \int \mathcal{D}\rho \mathcal{D}\Theta \mathcal{D}\mathbf{A}_{\mathcal{P}} \exp \frac{i}{\hbar} \left[S[\rho, \Theta, \mathbf{A}_{\mathcal{P}}] + \int d^4x (J\rho + J'\Theta + \mathbf{J} \cdot \mathbf{A}_{\mathcal{P}}) \right] \cdot \delta[\partial_\mu A_{\mathcal{P}}^\mu] . \quad (37)$$

Here, the integration measures $\mathcal{D}\rho = \Pi_x d\rho(x)$, $\mathcal{D}\mathbf{A}_{\mathcal{P}} = \Pi d\mathbf{A}_{\mathcal{P}}$, but the remaining measure $\mathcal{D}\Theta = \text{Det}\rho^2 \Pi_x d\Theta(x)$. The extra factor of $\text{Det}\rho^2$ can be seen to arise if one begins with the generating functional first expressed as functional integrals over a complex scalar field and its complex conjugate. Alternatively, one can obtain the configuration space functional integral starting with the functional integral over phase space. Integration over the momentum conjugate to Θ produces the same factor [9].

Indeed, it is a similar factor which has been interpreted in [3] as representative of a background spacetime which is *conformally* flat, rather than flat, with the 'radial' component of the Higgs field ρ playing the role of the conformal mode. In [2], a slightly different interpretation is given of this radial Higgs field as a *dilaton* field. Formally, there is indeed novelty in both interpretations. However, when perturbative effects are accounted for, at least at the one loop level, such interpretations appear somewhat tenuous, as we argue later in this subsection.

The effective action $\Gamma[\Phi]$ which is the generating functional for one particle irreducible diagrams (1PI), is generically defined as usual through the Legendre transformation

$$\begin{aligned} \Gamma[\Phi] &= W[\mathcal{J}] - \int d^4x \mathcal{J} \cdot \Phi \\ \Phi &= \frac{\delta W[\mathcal{J}]}{\delta \mathcal{J}} , \end{aligned} \quad (38)$$

where, we have collectively labeled all fields as Φ and the sources as \mathcal{J} , and $W[\mathcal{J}]$, we recall, is the generating functional of connected Green's functions. The task is to compute

$\Gamma[\Phi]$ to $\mathcal{O}(\hbar)$ with a view to eventually obtaining the one loop effective potential defined by the relation

$$V_{eff}(\phi_0) \equiv - \Gamma(\Phi)|_{\Phi=\phi_0} \left(\int d^4x \right)^{-1}, \quad (39)$$

where, ϕ_0 are spacetime independent. Observe that $V_{eff}(\phi_0)$ is the generating functional for 1PI graphs with vanishing external momenta. Even though the scalar potential is classically scale invariant, a mass scale is generated through renormalization in the quantum theory, which breaks this scale invariance. The effective potential may thus have a minimum away from the origin in ρ -space, defined in terms of the renormalization mass scale, which, in turn, relates to values of the dimensionless physical parameters of the theory (dimensional *transmutation* [7]).

Instead of evaluation of the functional integral over the Θ and $\mathbf{A}_{\mathcal{P}}$ fields, we make a change of basis to Θ and \mathbf{Y} via (35) and make use of the action (36) which is independent of Θ . The latter appears only in the constraint which now becomes a statement of non-transversality in spacetime of the \mathbf{Y} field. Θ can be simply integrated out, leaving behind a field-independent normalization which we set to unity. The integration over ρ involves a saddle-point approximation around a field ρ_c which may be called a ‘quantum’ field, since it is the solution of the classical ρ -equation of motion augmented by $\mathcal{O}(\hbar)$ corrections. With no gauge ambiguities anywhere, there is no question of gauge fixing; functional integration over the physical vector potential \mathbf{Y} can be performed straightforwardly.

Following ref. [8], the one loop effective action is given schematically by

$$\Gamma^{(1)}[\rho_c] = S[\rho_c, 0, 0] - i\hbar Z^{(1)}[\rho_c], \quad (40)$$

where,

$$Z^{(1)}[\rho_c] \equiv \int \mathcal{D}\rho \mathcal{D}\mathbf{Y} \exp \frac{i}{2\hbar} \left[\int d^4x d^4y \rho(x) \mathcal{M}_{\rho\rho}(x, y) \rho(y) + Y^\mu(x) \mathcal{M}_{Y_\mu Y_\nu}(x, y) Y^\nu(y) \right], \quad (41)$$

with, generically,

$$\mathcal{M}_{AB}(x, y) \equiv \left(\frac{\delta^2 S[\Phi]}{\delta \Phi_A(x) \delta \Phi_B(y)} \right)_{\Phi=\rho_c, 0, 0}. \quad (42)$$

Since our object of interest is the one loop effective potential, we restrict ourselves to a saddle point ρ_c which is spacetime independent. The matrices \mathcal{M} turn out to be diagonal in field space for the purpose of a one loop computation, with entries

$$\begin{aligned} \mathcal{M}_{\rho\rho} &= - \left(\square + \frac{\lambda}{2} \rho_c^2 \right) \delta^{(4)}(x - y) \\ \mathcal{M}_{Y_\mu Y_\nu} &= [\eta_{\mu\nu} (\square + e^2 \rho_c^2) - \partial_\mu \partial_\nu] \delta^{(4)}(x - y). \end{aligned} \quad (43)$$

One obtains easily

$$Z^{(1)}[\rho_c] = (\text{Det} [\mathcal{M}_{\rho\rho} \mathcal{M}_{YY}])^{-1/2} , \quad (44)$$

The functional determinants are evaluated in momentum space following [8], and one obtains for the one loop effective potential, using eq. (39), the expression

$$\begin{aligned} V_{eff}(\rho_c) &= \frac{1}{4!} \lambda \rho_c^4 + \hbar \int d^4k \log \left[(-k^2 + e^2 \rho_c^2)^{3/2} (-k^2 + \lambda \rho_c^2)^{1/2} \right] \\ &+ \frac{1}{2} B \rho_c^2 + \frac{1}{4!} C \rho_c^4 , \end{aligned} \quad (45)$$

where B and C are respectively the mass and coupling constant counterterms. The momentum integral is performed with a Lorentz-invariant cut-off $k^2 = \Lambda^2$, yielding

$$\begin{aligned} V_{eff}(\rho_c) &= \frac{1}{4!} \rho_c^4 + \frac{1}{2} B \rho_c^2 + \frac{1}{4!} C \rho_c^4 \\ &+ \frac{\hbar \rho_c^2 \Lambda^2}{32\pi^2} \left(\frac{1}{2} \lambda + 3e^2 \right) \\ &+ \frac{\hbar \rho_c^4}{64\pi^2} \left[\frac{1}{4} \lambda^2 \left(\log \frac{\lambda \rho_c^2}{2\Lambda^2} - \frac{1}{2} \right) + 3e^4 \left(\log \frac{e^2 \rho_c^2}{\Lambda^2} - \frac{1}{2} \right) \right] \end{aligned} \quad (46)$$

We remark here that in these manipulations, a $\exp(-\log \rho^2)$ term is generated in the one loop partition function, which cancels *exactly* against an identical term $\text{Det} \rho^2$ arising in the formal measure as discussed after eq. (37). This is precisely the point that was made earlier: the interpretation of that extra local factor in the formal functional measure as some sort of conformal mode in a conformally flat background cannot be sustained at the one loop level, since that factor is *eliminated* by a one loop contribution to the partition function. This has also been pointed out in ref. [10], but these authors attempt an alternate interpretation in terms of a ‘gauge-dependent gravity’ which we do not comment on here.

The mass and coupling constant renormalizations B and C are fixed through the renormalization conditions

$$\left. \frac{d^2 V}{d\rho_c^2} \right|_{\rho_c=M} = 0 \quad (47)$$

$$\left. \frac{d^4 V}{d\rho_c^4} \right|_{\rho_c=M} = \lambda \quad (48)$$

leading to the renormalized one loop effective potential

$$V_{eff}(\rho_c) = \frac{\lambda}{4!} \rho_c^4 + \rho_c^2 M^2 \left[-\frac{\lambda}{4} + \frac{9}{32\pi^2} (3e^4 + \frac{1}{2} \lambda^2) \right] \quad (49)$$

$$+ \left(\frac{3e^4}{64\pi^2} + \frac{\lambda^2}{256\pi^2} \right) \rho_c^4 \left[\log \frac{\rho_c^2}{M^2} - \frac{25}{6} \right] . \quad (50)$$

The potential has an extremum at $\rho_c = \langle \rho \rangle(M)$ leading eventually to the ratio of the squared masses of the Higgs boson to the photon

$$\frac{m_H^2}{m_A^2} = \frac{1}{e^2} \left[\frac{1}{3}\lambda - \left(\frac{3e^4}{8\pi^2} + \frac{\lambda^2}{32\pi^2} \right) \log \frac{\langle \rho \rangle^2}{M^2} - \frac{e^4}{\pi^2} - \frac{\lambda^2}{12\pi^2} \right] \quad (51)$$

The derivation of the mass ratio of the Higgs mass to the photon seemingly went through without any gauge fixing, since all fields being functionally integrated over are *physical* fields without any gauge ambiguity. The result (51) is thus a ‘physical’ result in this toy model where the photon acquires a mass. Notice that unlike in the original Coleman-Weinberg paper, we did not make an approximation of choosing $\lambda \sim e^4$, to drop terms of $O(\lambda^2)$. Thus, even though our result agrees with the earlier papers qualitatively, there are significant quantitative differences. However, the point in this section is not so much the result of the computation of the mass ratio, but the observation that the effect is physical and not a gauge artifact.

5 Linear Kalb-Ramond and graviton fields

The projection operator \mathcal{P} can be used to project out the physical (i.e., gauge invariant) part of the Kalb-Ramond antisymmetric second rank gauge potential [11] and the spin 2 linearized graviton field, as we now proceed to show.

5.1 Kalb-Ramond two form potential

The Kalb-Ramond two form potential \mathbf{B} has a field strength $\mathbf{H} = d\mathbf{B}$ which is clearly invariant under the gauge transformation $\mathbf{B} \rightarrow \mathbf{B} + d\Lambda$ for any one form field Λ . Construct now the projected two form field $\mathbf{B}^T \equiv \mathcal{P} \otimes \mathcal{P}\mathbf{B}$. Since, from eqn. (8) $\mathcal{P}df = 0 \ \forall f$, under the gauge transformation of \mathbf{B} , $\mathbf{B}^T \rightarrow \mathbf{B}^T + \mathcal{P} \otimes \mathcal{P}d\Lambda = \mathbf{B}^T$. Further, in a coordinate system,

$$\partial_\mu B^{T,\mu\nu} = 0 \quad (52)$$

implying that it is indeed transverse. Finally, it is clear that $\mathbf{H} = d\mathbf{B} = d\mathbf{B}^T$, which means that \mathbf{B}^T is indeed the physical part of the two form potential. As in the case of gauge free electrodynamics, one can formulate the theory of noninteracting Kalb-Ramond fields purely in terms of the physical gauge potential

$$S_{KR} = \frac{1}{2} \int d^4x \partial_\mu B_{\nu\rho\lambda}^T \partial^\mu B^{T,\nu\rho\lambda} \quad (53)$$

One can also augment the free action with an interaction with the gauge-free electrodynamics discussed in the earlier section

$$S_{KREM} = \int \mathbf{B}^T \wedge * \mathbf{F}(\mathbf{A}_{\mathcal{P}})$$

$$= \int \mathbf{H} \wedge \mathbf{A}_{\mathcal{P}} \quad (54)$$

which is manifestly invariant under both the $U(1)$ and KR gauge transformations.

5.2 Linearized graviton field

The graviton field is defined in terms of spin two fluctuations about Minkowski spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} . \quad (55)$$

If the Einstein-Hilbert action is expanded in powers of the spin 2 fluctuations $h_{\mu\nu}$ upto bilinear terms, the effective action is invariant under linearized infinitesimal coordinate (gauge) transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} . \quad (56)$$

Consider now the double projection on these spin 2 fluctuations

$$h_{\mu\nu}^T \equiv \mathcal{P}_\mu^\lambda \mathcal{P}_\nu^\rho h_{\lambda\rho} \quad (57)$$

with \mathcal{P} defined as earlier. It is easy to verify that under the linearized coordinate transformation (56), $h_{\mu\nu}^T$ is *invariant*. Further, it satisfies the spacetime transversality condition

$$\partial^\mu h_{\mu\nu}^T = 0 . \quad (58)$$

The linearized equation of motion for the graviton field is given by

$$\begin{aligned} \mathcal{G}_{\mu\nu} &\equiv \frac{1}{2}(\partial_\mu\partial_\nu h + \square h_{\mu\nu}) - \partial_\rho\partial_{(\mu}h_{\nu)}^\rho \\ &- \frac{1}{2}\eta_{\mu\nu}(\square h - \partial_\rho\partial_\lambda h^{\rho\lambda}) \\ &= 8\pi GT_{\mu\nu} , \end{aligned} \quad (59)$$

where, $h \equiv h^\mu_\mu$. In terms of the projected tensor field $h_{\mu\nu}^T$, this equation reduces to

$$\mathcal{G}_{\mu\nu} \equiv \frac{1}{2}\square(h_{\mu\nu}^T - \mathcal{P}_{\mu\nu}h^T) . \quad (60)$$

Defining

$$\bar{h}_{\mu\nu}^T \equiv h_{\mu\nu}^T - \mathcal{P}_{\mu\nu}h^T , \quad (61)$$

the linearized equation reduces to the inhomogeneous d'Alembert wave equation

$$\mathcal{G}_{\mu\nu} = \frac{1}{2}\square\bar{h}_{\mu\nu}^T = 8\pi GT_{\mu\nu} . \quad (62)$$

The field $\bar{h}_{\mu\nu}^T$ is also spacetime transverse and manifestly gauge invariant just like $h_{\mu\nu}^T$. One can show, following the procedure adopted for the photon field, that there is a Lorentz frame in which $\bar{h}_{\mu\nu}^T$ has only *two* spatial polarizations. Thus, this field can be used from the outset as a gauge free graviton field. Of course this is strictly valid for free spin 2 fields or at most in the interacting situation only perturbatively. As is well-known, it is harder to count polarizations for relatively stronger interacting situations.

6 Conclusion

The physical potential one form $\mathbf{A}_{\mathcal{P}}$ can be expressed in terms of the usual Maxwell gauge one form \mathbf{A} as

$$\mathbf{A}_{\mathcal{P}} = h^{-1}(x)(\mathbf{A} + d)h(x) = \mathbf{A} + h^{-1}dh , \quad (63)$$

where $h(x) \equiv \exp \int^x \mathbf{A}$. It is easy to see that

$$\begin{aligned} h^\omega(x) &\equiv e^{\int^x \mathbf{A}^\omega} = e^{\int^x \mathbf{A}} e^{\int^x d\omega} \\ &= h(x) \exp \omega(x) . \end{aligned} \quad (64)$$

Thus

$$\begin{aligned} \mathbf{A}_{\mathcal{P}}^\omega &= h^{-1} \exp(-\omega)[A - d\omega + d] \exp \omega h \\ &= \mathbf{A}_{\mathcal{P}} . \end{aligned} \quad (65)$$

This gives us the cue to attempt a construction of a physical *non-Abelian* one form in terms of the usual Yang-Mills gauge one form \mathbf{A} (which takes values in the Lie algebra of the gauge group) as

$$\mathbf{A}_{\mathcal{P}} = h^{-1}(\mathbf{A} + d)h \quad (66)$$

where, the Wilson lines $h \equiv \mathbf{P} \exp \int^x \mathbf{A}$, with \mathbf{P} denoting path ordering. If under a non-Abelian gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A}^\Omega = \Omega \mathbf{A} \Omega^{-1} - d\Omega \Omega^{-1} \quad (67)$$

one has $h^\Omega = \Omega h$, then once again, it is not difficult to show that $\mathbf{A}_{\mathcal{P}}^\Omega = \mathbf{A}_{\mathcal{P}}$. This is in a sense a generalization of the construction in [3] where the Wilson line is replaced by a group-valued field constructed out of the ‘phase’ components of the Higgs scalar, and transforms under gauge transformation precisely as h above. What we have not been determined yet is what constraint replaces the divergencefree condition (15) satisfied by $\mathbf{A}_{\mathcal{P}}$ in our Abelian case, so that the physics of non-Abelian potential one forms can be explored more thoroughly without gauge encumbrances. One also envisages application of these ideas to general relativity formulated as a gauge theory of Lorentz (or Poincaré) connection. We hope to discuss these and consequent issues elsewhere.

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References

- [1] W. Ehrenberg and R. E. Siday, Proc. Roy. Soc. **B62** (1949) 8; Y. Aharonov and D. Bohm, Phys. Rev. **115** (1959) 485.
- [2] M. N. Chernodub, L. D. Faddeev and A. Niemi, JHEP **0812** (2008) 014; arXiv:0804.1544 [hep-th].
- [3] L. D. Faddeev, arXiv:0811.3311 [hep-th].
- [4] G. 't Hooft, *Under the Spell of the Gauge Principle*, World Scientific Publishing Co. Pte. Ltd. (1994).
- [5] S. Weinberg, *Quantum Field Theory* Vol. 1 and 2, Cambridge University Press (1995).
- [6] S. Coleman, *Aspects of symmetry*, Cambridge University Press (1985).
- [7] S. Coleman and E. Weinberg, Phys. Rev. **D 7** (1973) 1888.
- [8] R. Jackiw, Phys. Rev. **D 9** (1974) 2276.
- [9] P. Senjanovic, Ann. Phys. **100** (1976) 227.
- [10] M. G. Ryskin and A. V. Shuvaev, arXiv:0909.3347 [hep-ph]
- [11] M. Kalb and P. Ramond Phys. Rev. **D 9** (1974) 2273.